MATH 162A Review: Inner Product

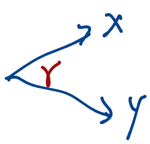
Facts to Know:

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Inner product, norm, and distance.

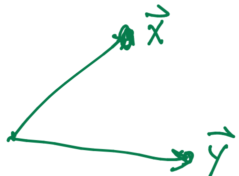
dot product

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$



$$x \cdot y = \|x\| \cdot \|y\| \cdot \cos \gamma$$ ← angle of two vectors.

$$\|x\| = \sqrt{x \cdot x}$$ ← length of x



$x - y = x + (-1) \cdot y$

$$\text{dist}(x, y) = \|x - y\| = \sqrt{(x-y) \cdot (x-y)}$$

Inner product

Definition. An inner product on a vector space V is a function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ satisfying the following

ⓐ $\langle x, y \rangle = \langle y, x \rangle$ symmetry

ⓑ $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$ linearity

ⓒ $\langle x, x \rangle \geq 0$, $\boxed{\langle x, x \rangle = 0 \Rightarrow x = 0}$

(positivity)

$\mathbb{R}^n$

$$\begin{cases} \langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n \\ \langle x, y \rangle = 2 x_1 y_1 + 3 x_2 y_2 + x_3 y_3 + \dots + x_n y_n \end{cases}$$

Examples:

$|\langle u, v\rangle|^2 \leq \|u\|^2 \cdot \|v\|^2$

1. Prove the Cauchy inequality

$$\cancel{|\langle u, v\rangle|^2 \leq \|u\|^2 \cdot \|v\|^2}.$$

$\cos\theta = \dfrac{\langle u, v\rangle}{\|u\| \cdot \|v\|}$

$u, v.$ $\quad w = u + tv$

$0 \leq \langle w, w\rangle = \langle u + tv, u + tv\rangle$

$= \langle u, u\rangle + \underline{\langle tv, tv\rangle} + \langle u, tv\rangle + \langle tu, v\rangle$

$= \|u\|^2 + t^2\|v\|^2 + 2t\langle u, v\rangle$

$f(t) = t^2\|v\|^2 + 2t\langle u, v\rangle + \|u\|^2 \geq 0$

$(2\langle u, v\rangle)^2 - 4\|v\|^2 \cdot \|u\|^2 \leq 0$

2. Let $\langle\ ,\ \rangle$ be an inner product of $\mathbb{R}^n$. Then there is a positive definite matrix A such that

$$\langle x, y\rangle = x^T A y.$$

$x^T A y$

Here x, y are column vectors of $\mathbb{R}^n$.

A is positive definite, if

① A is symmetric, $A^T = A$

② $\underline{x^T A x \geq 0}$. and if $x^T A x = 0 \Rightarrow x = 0$

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

$x = x_1 e_1 + \cdots + x_n e_n \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$y = y_1 e_1 + \cdots + y_n e_n$

$\langle x, y\rangle = \langle x_1 e_1 + \cdots + x_n e_n, y_1 e_1 + \cdots + y_n e_n\rangle$

$= \langle \sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\rangle = \underline{\sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle e_i, e_j\rangle}.$

$A = (\langle e_i, e_j\rangle)_{1 \leq i,j \leq n}$

$\langle x, x\rangle \geq 0 \qquad \sum_{i=1}^n \sum_{j=1}^n x_i x_j \langle e_i, e_j\rangle \geq 0 \Leftrightarrow x^T A x \geq 0$

if $\sum \sum x_i x_j \langle e_i, e_j\rangle = 0 \Rightarrow \langle x, x\rangle = 0, \Rightarrow x = 0.$